1. INTRODUCTION

Exact discrete-time models, which have been used extensively in the field of digital control, are well known for linear systems [1], whereas such models are not expected to exist for general nonlinear systems. However, it is very important to have a list of nonlinear systems for which such models are known and to enlarge this list. In this study, the exact discrete-time model means that their state matches exactly those of the continuous-time counterpart at the discrete-time instants for any sampling interval for certain types of inputs. In the previous papers [2-4], exact models have been considered for a class of important first-order nonlinear systems, such as those governed by the logistic, Bernoulli, and Riccati equations. The approach taken in these papers is to develop suitable variable transformations that can transform nonlinear systems into linear systems, for which the exact discrete-time models are known. Such a model is shown to be useful for the design of state observers [5] and feedback controllers [2, 6] as well. It should also be useful for discretizing analog control systems with good performance, such as the extension of the PIM method [7] to nonlinear systems.

The work of these previous papers has been extended to a class of second-order nonlinear system in [8]. Exact models are developed there for a class of second-order nonlinear systems, which consists of an independent first-order system whose variable affects the other first-order system that follows. In the present study, this technique is more generalized and shown as example cases. The first is for a higher power case of the above, the second is related to the stabilization of linearized systems, and the third is for the case where the use of emersion is suggested [9]. Simulation results are presented for the first and third cases to show that the exact discrete-time models are valid.

2. EXACT MODEL FOR LINEAR SYSTEMS

For a linear system given by
\[
\frac{dx}{dt} = Ax(t) + Bu(t),
\]
(1)
where \( u(t) \) is a piecewise-constant input, as in a system under sampled-data control with a zero-order-hold in front, the exact discrete-time model is given by
\[
\delta x_k = \Delta x \Gamma \Delta x_k + Bu_k ,
\]
(2)
where \( \delta = (q-1)/T \), \( q \) is the usual shift-left operator, and \( T \) is the sampling interval. The gain \( \Gamma \) is defined as
\[
\Gamma = \frac{1}{T} \int_0^T e^{\alpha t} dt ,
\]
(3)
which is constant for fixed \( A \) and \( T \), and approaches identity matrix as \( T \rightarrow 0 \). This gain adjusts its value such that the response \( x_k \) matches that of the continuous-time counterpart at discrete-time instants for any discretization interval. Any nonlinear system that can be converted into the linear form will have the exact discretization.

3. LINEARIZATION

Consider the following nonlinear system:
\[
\begin{align*}
\dot{x} &= a_x x^2 + a_1 x + a_0 = f_1, \\
\dot{y} &= (b_x x^2 + b_1 x + b_0)(c_2 y^2 + c_1 y + c_0) = f_2,
\end{align*}
\]
(4)
where it is assumed that
\[ a_1^2 - 4a_2a_0 \geq 0, \quad c_1^2 - 4c_2c_0 \geq 0. \] (5)
The first equation in Eq. (4) is the Riccati equation and
the second equation is affected by the first, but not the
other way. Introducing a new variable \( z \), with associated
parameters \( \epsilon_1 \) and \( \epsilon_2 \) to be determined shortly, by
\[ y = \frac{1}{z + \epsilon_1} + \epsilon_2 \] (6)
the second equation in (4) can be written as
\[ \dot{z} = -(b_2x^2 + b_1x + b_0)(z + \epsilon_1)^2 \times \left\{ c_2\left( \frac{1}{z + \epsilon_1} + \epsilon_2 \right)^2 + c_1\left( \frac{1}{z + \epsilon_1} + \epsilon_2 \right) + c_0 \right\} \]
\[ = -(b_2x^2 + b_1x + b_0) \times \left\{ c_2\epsilon_1^2 + c_1\epsilon_1^2 + c_0 \right\}(z + \epsilon_1)^2 \]
\[ + (2c_2\epsilon_2 + c_1)(z + \epsilon_1) + c_2 \}. \] (7)
Choosing \( \epsilon_2 \) such that
\[ c_2\epsilon_1^2 + c_1\epsilon_2 + c_0 = 0 \] (8)
is satisfied, equation (7) becomes
\[ \dot{z} = -(b_2x^2 + b_1x + b_0)\left( \sqrt{c_1^2 - 4c_2c_0}z + c_1\epsilon_1 + c_2 \right) \]
\[ = -(b_2x^2 + b_1x + b_0)\left\{ \sigma, z + \left( \sigma, \epsilon_1 \right) \right\} \] (9)
where
\[ \sigma = \sqrt{c_1^2 - 4c_2c_0} \]. (10)
In addition, if \( \epsilon_1 \) is chosen such that
\[ \sigma, \epsilon_1 + c_2 = 0 \], (11)
it follows that
\[ \dot{z} = -\sigma, (b_2x^2 + b_1x + b_0)z \]. (12)

For this type of nonlinear systems, a linearization has
been found in [8]. Using this method, a linearizing
variable transformation can be found as
\[ x = \frac{1}{w}, \quad y = \frac{1}{\sqrt{\alpha e^{(z-\lambda)}(x-\lambda)} + \eta} \] (13)
where \( \alpha \) is chosen such that the sign of \( \alpha^{-1}\sigma, \rho \) is
positive, but otherwise arbitrary, with
\[ \rho = \frac{1}{a_2} \left\{ \frac{a_1 + \sigma}{b_2} \left( a_1b_1 - a_2b_1 \right) + \left( a_2b_0 - b_2a_1 \right) \right\} \] (14)
and
\[ \sigma = \sqrt{a_1^2 - 4a_2a_0} \]. (15)
In the above, the linearization is possible when parameters are chosen as
\[ \beta = -\sigma, \frac{b_2}{a_2}, \quad \gamma = -\sigma, \frac{a_1b_1 - b_2a_1}{a_2^2}, \quad \lambda = -\sigma, \frac{a_1 + \sigma}{2a_2} \] (16)
and \( \epsilon \) and \( \eta \) to satisfy
\[ c_1\epsilon^2 + c_1\epsilon + c_0 = 0, \quad \sigma, \eta + c_2 = 0 \]. (17)
Under these conditions, the linearized diagonal system is obtained as
\[ \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\alpha^{-1}\sigma, \rho & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ -a_2 \end{bmatrix} \]. (18)

4. DISCRETIZATION

For the linearized system (18), the exact discrete-time
model can be calculated and the inverse variable
transformation carried out to obtain the following exact
discretization:
\[ \begin{cases} \delta x_k = \frac{\Gamma_\sigma}{1 - a_T T \sigma(x_k - \lambda)} f_i(x_k) = f_i, \\
\delta y_k = \Psi f_i(x_k), \end{cases} \] (19)
where
\[ \Gamma_\sigma = \frac{e^{\sigma T} - 1}{\sigma T} \Phi \]
\[ = -\sigma, c_T \Phi (y_k - \eta)(b_2x_k^2 + b_1x_k + b_0) \]
\[ \Phi = \left( \frac{e^{\sigma T}}{1 - a_T T \sigma(x_k - \lambda)} \right)^{\rho} \left( \frac{e^{\sigma T} - 1}{(b_2x_k^2 + b_1x_k + b_0) T} \right). \] (20)

As an example consider the following nonlinear
system:
\[ \begin{cases} \dot{x} = -x^2 + x + 2 \\ \dot{y} = (-x^2 + x + 1)(-y^2 + 2y + 1) \end{cases} \] (21)
where
\[ x(0) = 1.0, \quad y(0) = 0.5 \]. (22)
Simulations are carried out and shown in Figures 1 and
2, respectively, for \( T = 0.3 \) and 1.0 seconds. It can be
seen from these figures that the exact model gives the
response which matches exactly that of the
continuous-time original at the sampling instants for
both \( T \) values. The forward difference model, on the
other hand, does not coincide with the exact value, and becomes oscillatory for \( T = 1.0 \) second.

![Figure 1. Responses of the Continuous-Time, Forward Difference, and Exact Discrete-Time System for \( T = 0.3 \) second.](image)

Introducing the new variable as

\[ x = \frac{1}{u} \]  

(24)
eqn. (23) can be written as

\[ \dot{u} = -bu^2 - au \]  

(25)
Defining further a variable by

\[ u = v + \lambda \]  

(26)
the logistic equation (25) can be rearranged into

\[ \dot{v} = -bv^2 - (2b\lambda + a)v - \lambda(b\lambda + a) \]  

(27)
Thus, by choosing

\[ \lambda = -\frac{a}{b} \]  

(28)
eqn. (27) can be rearranged as

\[ \dot{v} = -bv^2 + av \]  

(29)
Note the sign change of the second terms in the right-hand-side of eqns. (25) and (29). Finally, using the transformation given by

\[ v = \frac{1}{w} \]  

(30)
eqn. (29) is converted into

\[ \dot{w} = -aw + b \]  

(31)
which is stable. Since

\[ x = \frac{w}{1 - \frac{a}{b}w} \]  

(32)
one can use the convergent variable \( w \) instead of the divergent variable \( x \). This idea is applicable in many situations, such as the following:

\[ \begin{align*}
 x &= axy, \\
y &= bxy
\end{align*} \]  

(33)
with

\[ x(0) = x_0, y(0) = y_0. \]  

(34)

This is a case where the first and second variables interfere each other, unlike the system (4). Using the conventional transform of \( x = 1/w \) and \( y = 1/v \), system (33) is transformed into

\[ \begin{align*}
 \dot{w} &= (bx_0 - ay_0)w - b \\
\dot{v} &= -(bx_0 - ay_0)v - a,
\end{align*} \]  

(35)
which cannot be stable. However, using the method explained earlier, this system is transformed into

\[ \begin{align*}
 \dot{w} &= (bx_0 - ay_0)w - b \\
\dot{z} &= (bx_0 - ay_0)z - a,
\end{align*} \]  

(36)
which can be made stable.

5. Extension 1

It has recently been reported that many nonlinear systems encountered in engineering applications can be expressed as a simpler, but generally higher order, system using the concept of immersion [9]. This may be a useful technique to convert nonlinear systems into one of the forms expressible by (4). As an example, consider the following system under sampled-data control:

$$\dot{\theta}(t) = \{A \sin \omega \theta(t)\} u(t).$$  (37)

with $u(t)$ constant between the sampling interval. By choosing the variables as

$$x = \cos \omega \theta, \quad y = \sin \omega \theta,$$

eqn. (37) can be rewritten as

$$\begin{cases}
\dot{x} = \omega A(x^2 - 1)u \\
\dot{y} = \omega Axyu
\end{cases}$$  (39)

which is a special case of (4) where $c_3 = c_2 = 0$. This gives the following parameters:

$$\varepsilon = \eta = 0, \quad \sigma = 2\omega A, \quad \sigma_r = 1, \quad \rho = \omega A.$$  (40)

In fact, using the transformation given by

$$\begin{cases}
x = \frac{1}{w} + \lambda \\
y = v^{-\alpha} e^{-\beta(x-\lambda)\gamma}
\end{cases}$$  (41)

where

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 1, \quad \lambda = 1,$$

eq. (37) can be linearized exactly as

$$\begin{bmatrix}
\dot{v} \\
\dot{w}
\end{bmatrix} = 
\begin{bmatrix}
-\omega A & 0 \\
0 & -2\omega A
\end{bmatrix}
\begin{bmatrix}
v \\
w
\end{bmatrix} + 
\begin{bmatrix}
0 \\
-\omega A
\end{bmatrix}.$$  (43)

Exactly discretizing this linear model, and converting it back into the nonlinear form with variable $x$ and $y$, the following model can be obtained:

$$\begin{cases}
\delta x_i &= \frac{\Gamma_{2nd} e^{\omega A T \Gamma_{2nd}^{-1}\omega A(x_i^2 - 1)}}{1 - \omega A T \Gamma_{2nd}^{-1}\omega A(x_i^2 - 1)} = f_x, \\
\delta y_i &= \frac{\Gamma_{2nd}}{1 - \omega A T \Gamma_{2nd}^{-1} - \omega A x_i T} - \omega A x_i y_i = f_y,
\end{cases}$$  (44)

where

$$\Gamma_{2nd} = \frac{e^{2\omega A T} - 1}{2\omega A T}.$$  (45)

Expressed in the original variable $\theta$, the exact discrete-time model is obtained as

$$\delta \theta_k = \frac{1}{\omega} \cos^{-1} f_x = \frac{1}{\omega} \sin^{-1} f_y.$$  (46)

Simulations are carried out for the case of $\omega = 3$, $A = 1$, and $\theta_0 = 0.1$, and the results are shown in Figures 3 and 4 for $T = 0.3$ and 1.0 seconds, respectively.

![Figure 3](image3.png)

Figure 3. Theta responses of the Continuous-Time, Forward Difference, and Exact Discrete-Time System for $T = 0.3$ second.

![Figure 4](image4.png)

Figure 4. Theta responses of the Continuous-Time, Forward Difference, and Exact Discrete-Time System for $T = 1.0$ second.

As in Figures 1 and 2, the state $\theta_k$ of the derived exact model matches that of the continuous-time system at the sampling instants for all $T$ values. Although the state responses of both discrete-time systems approach that of the continuous-time system as the sampling interval approaches zero, the performance of the standard forward difference method quickly deteriorates as the sampling interval increases and often becomes highly oscillatory.
6. EXTENSION 2

Consider the third-order system given by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= ax_3^2,
\end{align*}
\]

(47)

which is in a normal form [11]. From numerous insights acquired from past experiences for first and second order cases, the following set of new variables is obtained for linearization:

\[
\begin{align*}
x_1 &= \log v_1 + \frac{1}{2a\alpha}(\log v_2)^2 \\
&\quad + \frac{1}{a^2}v_3(\log v_3 - 1) \\
&\quad - \frac{\alpha}{2a}v_3^2 + \beta v_3 \\
x_2 &= \log v_2 v_3^\alpha + \alpha v_3 \\
x_3 &= \frac{1}{v_3},
\end{align*}
\]

(48)

where \(\alpha\) and \(\beta\) are free parameters. Using the above variable transformations, system (47) can be converted into a linear system given by

\[
\begin{align*}
\dot{v}_1 &= a\beta v_1 \\
\dot{v}_2 &= a\alpha^2 v_2 \\
\dot{v}_3 &= -a
\end{align*}
\]

(49)

which can be made stable using the free parameters if necessary. The exact discrete-time model of this system is known and can be converted into the one in the original variables, as

\[
\begin{align*}
\delta x_{1k} &= x_{2k} + \frac{1}{a} + \frac{1-aT_x}{a^2 T_{x_{1k}}} \log(1-aT_{x_{1k}}) \\
\delta x_{2k} &= -\frac{\log(1-aT_{x_{1k}})}{aT} \\
\delta x_{3k} &= \frac{a}{1-aT_{x_{1k}}} x_{3k}^2.
\end{align*}
\]

(50)

Simulation results are shown in Figures 5, 6, and 7 for the case of \(a = -0.8\) with all the initial conditions for the state variables being set to 1.0. The sampling interval is chosen to be \(T=1.0\) second. It can be seen again, that all the state variables of the exact discrete-time model gives the values identical to those of the continuous-time system at the sampling instants.
7. CONCLUSION

Presented in this paper is a way to obtain the exact discrete-time model for a class of nonlinear systems where the first state affects the second state and not the other way. Unlike the conventional difference models and invariant type models, the proposed exact discrete-time models give response that match those of the continuous-time original at the sampling instants for any sampling intervals.

It is believed that, although the approach is not applicable to a general class of nonlinear systems, it is nevertheless very important to widen the class of nonlinear systems for which exact discrete-time models can be found. One approach along this line is to incorporate the concept of immersion [9], as demonstrated in the present paper. A further study is desired on this topic.

Extension to simultaneous case with non diagonal interactions to third and higher order systems is underway. One such example has been presented for a third-order system expressed in a normal form.

Acknowledgement

The work presented in this paper has been supported by JSPS grant #17560202.

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